PROGRESSIONS

(KEY CONCEPTS + SOLVED EXAMPLES)

PROGRESSIONS

1.Arithmetic Progression (A.P.)

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KEY CONCEPTS

1. Definition

When the terms of a sequence or series are arranged under a definite rule then they are said to be in a Progression. Progression can be classified into 5 parts as-

(i) Arithmetic Progression (A.P.)

(ii) Geometric Progression (G.P.)

(iii) Arithmetic Geometric Progression (A.G.P.)

(iv) Harmonic Progression (H.P.)

(v) Miscellaneous Progression

2. Arithmetic Progression (A.P.)

Arithmetic Progression is defined as a series in which difference between any two consecutive terms is constant throughout the series. This constant difference is called common difference If 'a' is the first term and 'd' is the common difference, then an AP can be written as

$a + (a+ d) + (a + 2d) + (a + 3d) + ...$

Note: If a,b,c, are in AP $\Leftrightarrow 2b = a + c$

2.1 General Term of an AP General term (n^{th} term) of an AP is given by

$T_n = a + (n-1)\,d$

Note: (i) General term is also denoted by ℓ (last term)

(ii) n (No. of terms) always belongs to set of natural numbers.

(iii) Common difference can be zero, + ve or – ve.

(iv) n^{th} term from end is given by

$= T_m - (n-1)\,d$

or $= (m - n + 1)^{th}$ term from beginning where m is total no. of terms.

2.2 Sum of n terms of an A.P.

The sum of first n terms of an A.P. is given by

$S_n = \frac{n}{2}[2a + (n-1)d]$ or $S_n = \frac{n}{2}[a + T_n]$

Note: (i) If sum of n terms S_n is given then general term $T_n = S_n - S_{n-1}$ where S_{n-1} is sum of (n–1) terms of A.P.

(ii) Common difference of AP is given by $d = S_2 - 2S_1$ where S_2 is sum of first two terms and S_1 is sum of first term or first term.

(iii) The sum of infinite terms of an A.P. is ∞ if $d > 0$ and $-\infty$ if $d< 0$.

(iv) Sum of n terms of an A.P. is of the form $An^2+ Bn$ i.e. a quadratic expression in n, in such case the common difference is twice the coefficient of n^2. i.e. 2A

(v) n^{th} term of an A.P. is of the form An + B i.e. a linear expression in n, in such a case the coefficient of n is the common difference of the A.P. i.e. A

(vi) If for the different A.P's

$$\frac{S_n}{S'_n} = \frac{f_n}{\phi_n} \text{ then } \frac{T_n}{T'_n} = \frac{f(2n-1)}{\phi(2n-1)}$$

(vii) If for two A.P.'s $\frac{T_n}{T'_n} = \frac{An+B}{Cn+D}$

$$\text{then } \frac{S_n}{S'_n} = \frac{A\left(\frac{n+1}{2}\right)+B}{C\left(\frac{n+1}{2}\right)+D}$$

3. Arithmetic Mean (A.M.)

If three or more than three terms are in A.P., then the numbers lying between first and last term are known as Arithmetic Means between them.i.e. The A.M. between the two given quantities a and b is A so that a, A, b are in A.P.

i.e. $A - a = b - A \Rightarrow A = \frac{a+b}{2}$

Note : A.M. of any n positive numbers $a_1, a_2 ......a_n$ is

$$A = \frac{a_1 + a_2 + a_3 + ...a_n}{n}$$

3.1 n AM's between two given numbers

If in between two numbers 'a' and 'b' we have to insert n AM $A_1, A_2, ...A_n$ then a, $A_1, A_2, A_3....A_n$, b will be in A.P. The series consist of (n+2) terms and the last term is b and first term is a.

$\Rightarrow a + (n + 2 - 1)\,d = b$

$\Rightarrow d = \frac{b-a}{n+1}$

$A_1 = a + d$, $A_2 = a + 2d$,.... $A_n = a + nd$ or $A_n = b - d$

Note: (i) Sum of n AM's inserted between a and b is equal to n times the single AM between a and b i.e. $\sum_{r=1}^{n} A_r = nA$ where $A = \frac{a+b}{2}$

(ii) between two numbers

$$\frac{\text{sum of m AM's}}{\text{sum of n AM's}} = \frac{m}{n}$$

4. Supposition of Terms A. P.

(i) When no. of terms be odd then we take

three terms are as: a – d, a, a+ d

five terms are as– 2d, a – d, a, a+ d, a + 2d

Here we take middle term as 'a' and common difference as 'd'.

(ii) When no. of terms be even then we take

4 term are as : a – 3d, a– d, a + d, a + 3d

6 term are as = a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d

Here we take 'a – d, a + d' as middle terms and common difference as '2d'.

Note: (i) If no. of terms in any series is odd then only one middle term is exist which is $\left(\frac{n+1}{2}\right)^{th}$ term where n is odd.

(ii) If no. of terms in any series is even then middle terms are two which are given by

$(n/2)^{th}$ and $\left\{\left(\frac{n}{2}\right)+1\right\}^{th}$ term where n is even.

5. Some Properties of an A. P.

(i) If each term of a given A.P. be increased, decreased, multiplied or divided by some non zero constant number then resulting series thus obtained will also be in A.P.

(ii) In an A.P., the sum of terms equidistant from the beginning and end is constant and equal to the sum of first and last term.

(iii) Any term of an AP (except the first term) is equal to the half of the sum of terms equidistant from the term i.e.

$$a_n = \frac{1}{2}(a_{n-k} + a_{n+k}),\ k < n$$

(iv) If in a finite AP, the number of terms be odd, then its middle term is the AM between the first and last term and its sum is equal to the product of middle term and no. of terms.

5.1 Some standard results

(i) Sum of first n natural numbers

$$\Rightarrow \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

(ii) Sum of first n odd natural numbers

$$\Rightarrow \sum_{r=1}^{n} (2r-1) = n^2$$

(iii) Sum of first n even natural numbers

$$\Rightarrow \sum_{r=1}^{n} 2r = n(n+1)$$

(iv) Sum of squares of first n natural numbers

$$\Rightarrow \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

(v) Sum of cubes of first n natural numbers

$$\Rightarrow \sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$$

(vi) If r^{th} term of an A.P.

$T_r = Ar^3 + Br^2 + Cr + D$, then

sum of n term of AP is

$$S_n = \sum_{r=1}^{n} T_r = A\sum_{r=1}^{n} r^3 + B\sum_{r=1}^{n} r^2 + C\sum_{r=1}^{n} r + D\sum_{r=1}^{n} 1$$

(vii) If for an A.P. p^{th} term is q, q^{th} term is p then m^{th} term is = p + q – m

(viii) If for an AP sum of p terms is q, sum of q terms is p, then sum of (p + q) term is – (p + q).

(ix) If for an A.P. sum of p terms is equal to sum of q terms then sum of (p + q) terms is zero.

6. Geometrical Progression (G. P.)

Geometric Progression is defined as a series in which ratio between any two consecutive terms is constant throughout the series. This constant ratio is called as a common ratio. If 'a' is the first term and 'r' is the common ratio, then a GP can be written as $a + ar + ar^2 + ...$

Note : a, b, c are in G.P. if $\Leftrightarrow b^2 = ac$

6.1 General Term of a G.P. :

General term (n^{th} term) of a G.P. is given by

$T_n = ar^{n-1}$

Note : (i) n^{th} term from end is given by $= \frac{T_m}{r^{n-1}}$ where m stands for total no. of terms

(ii) If a_1, a_2, a_3 ..... are in GP, then

$$r = \left(\frac{a_k}{a_p}\right)^{\frac{1}{k-P}}$$

6.2 Sum of n terms of a G.P.

The sum of first n terms of an A.P. is given by

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a - rT_n}{1-r} \quad \text{when } r < 1$$

or $$S_n = \frac{a(r^n-1)}{r-1} = \frac{rT_n - a}{r-1} \quad \text{when } r > 1$$

and $S_n = an$ when $r = 1$

6.3 Sum of an infinite G.P.

The sum of an infinite G.P. with first term a and common ratio r ($-1 < r < 1$ i.e. $|r| < 1$) is

$$S_\infty = \frac{a}{1-r}$$

Note : If $r \geq 1$ then $S_\infty \to \infty$

7. Geometrical Mean (G. M.)

If three or more than three terms are in G.P. then all the numbers lying between first and last term are called Geometrical Means between them. i.e.

The G.M. between two given quantities a and b is G, so that a, G, b, are in G.P.

i.e. $\frac{G}{a} = \frac{b}{G} \Rightarrow G^2 = ab \quad G = \sqrt{ab}$

Note : (i) G.M. of any n positive numbers $a_1, a_2, a_3, \ldots.. a_n$ is $(a_1.a_2.a_3.....a_n)^{1/n}$.

(ii) If a and b are two numbers of opposite signs, then GM between them does not exist.

7.1 n GM's between two given numbers

If in between two numbers 'a' and 'b', we have to insert n GM $G_1, G_2....G_n$ then a, $G_1, G_2, \ldots. G_n$, b will be in GP. The series consist of (n+2) terms and the last term is b and first term is a.

$$\Rightarrow ar^{n+2-1} = b \qquad \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$G_1 = ar, G_2 = ar^2 ......G_n = ar^n$ or $G_n = b/r$

Note : Product of n GM's inserted between 'a' and 'b' is equal to n^{th} power of the single GM between 'a' and 'b'

i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where $G = \sqrt{ab}$

8. Supposition of Terms in a G. P.

(i) When no. of term be odd.

then we take three terms as a/r, a, ar

Five terms as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

Here we take middle term as 'a' and common ratio as 'r'.

(ii) When no. of terms be even then we take

4 terms as : $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

6 terms as : $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$

Here we take $\frac{a}{r}$, ar as middle terms and common ratio as r^2.

9. Some Properties of a G.P.

(i) If each term of a G.P. be multiplied or divided by the same non zero quantity, then resulting series is also a G.P.

(ii) In an G.P., the product of two terms which are at equidistant from the first and the last term, is constant and is equal to product of first and last term.

(iii) If each term of a G.P. be raised to the same power, then resulting series is also a G.P.

(iv) In a G.P. every term (except first) is GM of its two terms which are at equidistant from it.

i.e. $T_r = \sqrt{T_{r-k}T_{r+k}} \quad k < r$

(v) In a finite G.P. , the number of terms be odd then its middle term is the G.M. of the first and last term.

(vi) If the terms of a given G.P. are chosen at regular intervals, then the new sequence is also a G.P.

(vii) If $a_1, a_2, a_3 ....a_n$ is a G.P. of non zero, non negative terms, then $\log a_1, \log a_2, \ldots\ldots \log a_n$ is an A.P. and vice-versa.

(viii) If a_1, a_2, a_3....... and b_1, b_2, b_3 .... are two G.P.'s then $a_1 b_1, a_2b_2, a_3b_3$..... is also in G.P.

9.1 Recurring Decimals : To find the fractional value of a decimal number whose decimal part contains a repetition of digits we use G.P. This concept will be more clear by the examples.

10. Arithmetico-Geometrical Progression (A.G.P.)

If each term of a Progression is the product of the corresponding terms of an A.P. and a G.P., then it is called arithmetic- geometric progression (A. G.P.)

e.g. a, (a + d) r, (a + 2d) r^2 , ...

The general term (n^{th} term) of an A.G.P. is

$T_n = [a + (n-1) d] r^{n-1}$

To find the sum of n terms of an A.G.P. we suppose its sum S, multiply both sides by the common ratio of the corresponding G.P. and than subtract as in following way and we get a G.P. whose sum can be easily obtained.

$S_n = a + (a + d) r + (a + 2d) r^2 + ...[a + (n-1) d] r^{n-1}$

$rS_n = ar + (a+d) r^2 + ...+ [a+(n-1) d] r^n$

After subtraction we get

$S_n (1- r) = a+ r.d + r^2 d +...dr^{n-1}- [a + (n-1)d] r^n$

After solving

$$S_n = \frac{a}{1-r} + \frac{r.d(1-r^{n-1})}{(1-r)^2}$$

and $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

Note : This is not a standard formula. This is only to understand the procedure for finding the sum of an A.G.P. However formula for sum of infinite terms can be used directly.

11. Harmonical Progression (H.P.)

Harmonical Progression is defined as a series in which reciprocal of its terms are in A.P. The standard form of a H.P. is

$$\frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + ...$$

Note : a, b, c are in H.P. $\Leftrightarrow$ $b = \frac{2ac}{a+c}$

11.1 General Term of a H.P.

General term (n^{th} term) of a H.P. is given by

$$T_n = \frac{1}{a+(n-1)d}$$

Note : (i) There is no formula and procedure for finding the sum of H.P.

(ii) If a, b, c are in H.P. then $\frac{a}{c} = \frac{a-b}{b-c}$

11.2 Harmonical Mean (H.M.)

If three or more than three terms are in H.P., then all the numbers lying between first and last term are called Harmonical Means between them. i.e.

The H.M. between two given quantities a and b is H so that a, H , b are in H.P.

i.e. $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P.

$$\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H} \Rightarrow H = \frac{2ab}{a+b}$$

11.2.1 n H. M's between two given numbers

To find n HM's between a, and b we first find n AM's between 1/a and 1/b then their reciprocals will be required HM's.

12. Relation between A.M., G.M. & H.M.

A, G, H are AM, GM and HM respectively between two numbers 'a' and 'b' then

$$A = \frac{a+b}{2}, G = \sqrt{ab}, H = \frac{2ab}{a+b}$$

(i) Consider $A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0$

So $A \geq G$

In the same way $G \geq H \Rightarrow A \geq G \geq H$

(ii) Consider $A.H = \frac{a+b}{2} . \frac{2ab}{a+b} = ab = G^2$

$\Rightarrow G^2 = A.H.$

13. Some Important Results

(i) If number of terms in an A.P./G.P./H.P. is odd then its mid term is the A.M./G.M./H.M. between the first and last number.

(ii) If the number of terms in an A.P./G.P./H.P. is even then A.M./G.M./H.M. of its two middle terms is equal to the A.M./G.M./H.M. between the first and last numbers.

(iii) a, b, c are in A.P. and H.P. $\Rightarrow$ a, b, c are in G.P.

(iv) If a, b, c are in A.P. then $\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}$ are in A.P.

(v) If a^2, b^2, c^2 are in A.P. then $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

(vi) If a, b, c are in G.P. then a^2, b^2, c^2 are in G.P.

(vii) If a, b, c, d are in G.P. then a + b, b + c, c + d are in G.P.

(viii) If a, b, c are in H.P. then $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P.

SOLVED EXAMPLE

Ex.1 If for an A.P. $T_3 = 18$ and $T_7 = 30$ then S_{17} is equal to-

(A) 612 (B) 622

(C) 306 (D) None of these

Sol. Let first term = a, common difference = d

Then $T_3 = a + 2d = 18$ and $T_7 = a + 6d = 30$

Solving these, a= 12, d= 3

$$\therefore S_{17} = \frac{17}{2}[2a + (17-1)d]$$

$$= \frac{17}{2}[24 + 16 \times 3] = 612$$

Ans. [A]

Ex.2 The first, second and middle terms of an AP are a, b, c respectively. Their sum is-

(A) $\frac{2(c-a)}{b-a}$ (B) $\frac{2c(c-a)}{b-a} + c$

(C) $\frac{2c(b-a)}{c-a}$ (D) $\frac{2b(c-a)}{b-a}$

Sol. We have first term = a, second term = b

$\therefore$ d = common difference = b – a

It is given that the middle term is c. This means that there are an odd number of terms in the AP. Let there be (2n+1) terms in the AP. Then $(n+1)^{th}$ term is the middle term.

$\therefore$ middle term = c $\Rightarrow$ a + nd = c

$$\Rightarrow a + n(b-a) = c \Rightarrow n = \frac{c-a}{b-a}$$

$$\therefore \text{Sum} = \frac{2n+1}{2}[2a + (2n+1-1)d]$$

$$= \frac{1}{2}\left\{2\left(\frac{c-a}{b-a}\right)+1\right\}\left[2a+2\left(\frac{c-a}{b-a}\right)(b-a)\right]$$

$$= \frac{1}{2}\left\{\frac{2(c-a)}{b-a}+1\right\}\{2c\} = \frac{2c(c-a)}{b-a} + c$$

Ans. [B]

Ex.3 The sum of integers in between 1 and 100 which are divisible by 2 or 5 is-

(A) 3100 (B) 3600

(C) 3050 (D) 3500

Sol. Required sum = (sum of integers divisible by 2) + (sum of integers divisible by 5) – (sum of integers divisible by 2 and 5)

= (2+4+6+....+100)+(5+10+15+.....+ 100) – (10 + 20 +....+ 100)

$$= \frac{50}{2}[2 \times 2 + (50-1) \times 2] + \frac{20}{2}[2\times5 + (20-1)\times10] - \frac{10}{2}[2\times10+(10-1)\times 10]$$

= 50 [2 + 49] + 10 [10 + 95] – 5 [20 + 90]

= 51 × 50 + 105 × 10 – 110 × 5 = 3050

Ans. [C]

Ex.4 If $a_1, a_2, a_3.....a_n$ are in AP where $a_i > 0 > i$ then the value of

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + ...+ \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} =$$

(A) $\frac{1}{\sqrt{a_1}+\sqrt{a_n}}$ (B) $\frac{1}{\sqrt{a_1}-\sqrt{a_n}}$

(C) $\frac{n}{\sqrt{a_1}-\sqrt{a_n}}$ (D) $\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$

Sol. Let d be the c.d. of the A.P. Now L.H.S.

$$= \frac{\sqrt{a_1}-\sqrt{a_2}}{a_1-a_2} + \frac{\sqrt{a_2}-\sqrt{a_3}}{a_2-a_3} + ... + \frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{a_{n-1}-a_n} \quad \text{(Note)}$$

$$= -\left(\frac{\sqrt{a_1}-\sqrt{a_2}+\sqrt{a_2}-\sqrt{a_3}+...\sqrt{a_n}}{d}\right)$$

$$= -\frac{(\sqrt{a_1}-\sqrt{a_n})}{d} = \frac{1}{d}\frac{(a_n-a_1)}{\sqrt{a_n}+\sqrt{a_1}}$$

$$= \frac{(n-1)d}{d[\sqrt{a_n}+\sqrt{a_1}]} \quad [\because a_n = a_1 + (n-1)d]$$

$$= \frac{n-1}{\sqrt{a_n}+\sqrt{a_1}}$$ **Ans.**

[D]

Ex.5 If a^2, b^2, c^2 are in A.P. then

$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+d}$ are in-

(A) A.P. (B) G.P.

(C) H.P. (D) None of these

Sol. $\because$ a^2, b^2, c^2 are in A.P.

$\therefore$ $a^2 + ab + bc + ca, b^2 + bc + ca + ab, c^2 + ca + ab + bc$ ..... are also in A.P.

[adding ab + bc + ca]

or $(a + c)(a + b), (b + c)(a + b), (c + a)(b + c)$ .. are also in A.P.

or $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+d}$ are in A.P.

[dividing by (a + b) (b + c) (c + a)]

Ans. [A]

Ex.6 If the sum of first 6 terms of a G.P. is nine times of the sum of its first three terms, then its common ratio is-

(A) 1 (B) 3/2 (C) 2 (D) – 2

Sol. $\frac{a(1-r^6)}{1-r} = 9\,\frac{a(1-r^3)}{1-r}$

$\Rightarrow$ $1 - r^6 = 9(1 - r^3)$ $(\because r \neq 1)$

$\Rightarrow$ $1 + r^3 = 9$

$\therefore$ $r = 2$ **Ans. [C]**

Ex.7 If p^{th}, q^{th} and r^{th} terms of an A.P. are equal to corresponding terms of a G.P. and these terms are respectively x, y, z, then

$x^{y-z} . y^{z-x} . z^{x-y}$ equals-

(A) 0 (B) 1

(C) 2 (D) None of these

Sol. Let first term of an A.P. be a and c.d. be d and first term of a G. P. be A and c.r. be R, then

$a + (p-1)d = AR^{p-1} = x$

$\Rightarrow$ $p - 1 = (x - a)/d$ ...(1)

$a + (q-1)d = AR^{q-1} = y$

$\Rightarrow$ $q - 1 = (y - a)/d$ ...(2)

$a + (r-1)d = AR^{r-1} = z$

$\Rightarrow$ $r - 1 = (z - a)/d$ ...(3)

$\therefore$ Given expression

$= (AR^{p-1})^{y-z}, (AR^{q-1})^{z-x}, (AR^{r-1})^{x-y}$

$= A^0\, R^{(p-1)(y-z)+(q-1)(z-x)+(r-1)+(x-y)}$

$= A^0 R^{[(x-a)(y-z)+(y-a)(z-x)+(z-a)(x-y)]/d}$

[By (1), (2) and (3)]

$= A^0 R^0 = 1$ **Ans. [B]**

Ex.8 If x, y, z are in G.P. and $a^x = b^y = c^z$ then-

(A) $\log_b a = \log_a c$ (B) $\log_c b = \log_a c$

(C) $\log_b a = \log_c b$ (D) None of these

Sol. x, y, z are in G.P. $\Rightarrow$ $y^2 = xz$ ...(i)

We have, $a^x = b^y = c^z = \lambda$ (say)

$\Rightarrow$ $x \log a = y \log b = z \log c = \log \lambda$

$\Rightarrow$ $x = \frac{\log \lambda}{\log a}, y = \frac{\log \lambda}{\log b}, z = \frac{\log \lambda}{\log c}$

putting x, y, z in (i) , we get

$\left(\frac{\log \lambda}{\log b}\right)^2 = \frac{\log \lambda}{\log a} \cdot \frac{\log \lambda}{\log c}$

$(\log b)^2 = \log a . \log c$

or $\log_a b = \log_b c \Rightarrow \log_b a = \log_c b$

Ans. [C]

Ex.9 If a, b, c, d are in G.P., then

$(a^3 + b^3)^{-1}, (b^3 + c^3)^{-1}, (c^3 + d^3)^{-1}$ are in-

(A) A.P. (B) G.P.

(C) H.P. (D) None of these

Sol. Let $b = ar$, $c = ar^2$ and $d = ar^3$. Then,

$\frac{1}{a^3+b^3} = \frac{1}{a^3(1+r^3)}, \frac{1}{b^3+c^3} = \frac{1}{a^3r^3(1+r^3)}$

and, $\frac{1}{c^3+d^3} = \frac{1}{a^3r^3(1+r^3)}$

Clearly, $(a^3 + b^3)^{-1}, (b^3 + c^3)^{-1}$ and $(c^3 + d^3)^{-1}$ are in G.P. with common ratio $1/r^3$.

Ans. [B]

Ex.10 If a, b, c, d and p are distinct real numbers such that $(a^2 + b^2 + c^2)p^2 - 2p(ab + bc + cd) + (b^2 + c^2 + d^2) \leq 0$ then a, b, c, d are in -

(A) A.P. (B) G.P.

(C) H.P. (D) None of these

Sol. Here the given condition

$(a^2 + b^2 + c^2)p^2 - 2p(ab + bc + ca) + b^2 + c^2 + d^2 \leq 0$

$\Rightarrow$ $(ap - b)^2 + (bp - c)^2 + (cp - d)^2 \leq 0$

Since the squares can not be negative

$\therefore$ ap – b = 0, bp – c = 0, cp – d = 0

$\Rightarrow \frac{1}{p} = \frac{a}{b} = \frac{b}{c} = \frac{c}{d}$

$\therefore$ a, b, c, d are in G.P. **Ans. [B]**

Ex.11 If p^{th}, q^{th} and r^{th} terms of H.P. are u, v, w respectively, then the value of the expression (q – r)vw + (r – p)wu + (p – q)uv is-

(A) 1 (B) 0 (C) – 2 (D) – 1

Sol. Let H.P. be

$\frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + ...$

$u = \frac{1}{a+(p-1)d}$, $v = \frac{1}{a+(q-1)d}$,

$w = \frac{1}{a+(r-1)d}$

$a + (p-1)d = \frac{1}{u}$, $a + (q-1)d = \frac{1}{v}$,

$a + (r-1)d = \frac{1}{w}$

$\Rightarrow (q-r)\{a + (p-1)d\} + (r-p)\{a + (q-1)d\} + ... = \frac{1}{u}(q-r) + \frac{1}{v}(r-p) + ...$

$\Rightarrow (q-r)vw + ... = 0$ **Ans. [B]**

Ex.12 If x,y,z are in A.P. and x,y, t are in G.P. then x, x– y, t– z are in -

(A) G.P. (B) A.P.

(D) H.P. (D) A.P. and G.P. both

Sol. x,y,z are in A.P.

$\Rightarrow$ 2y = x + z

or $2xy = x^2 + xz$ (multiplying with x)

$\Rightarrow x^2 - 2xy = -xz$ ...(1)

x,y, t are in G.P.

$\Rightarrow y^2 = xt$ ...(2)

or $(x^2 - 2xy + y^2) = -xz + xt$

or $(x-y)^2 = x(t-z)$

x, x – y, t– z are in G.P. **Ans. [A]**

Ex.13 The sum of the series

$a - (a+d) + (a+2d) - (a+3d) + ...$ upto $(2n+1)$ terms is-

(A) – nd (B) a + 2 nd

(C) a + nd (D) 2nd

Sol. The given series is an A.G.P. with common ratio

S = a – (a + d) + (a + 2d) – (a + 3d)+....+ (a + 2nd)

$\Rightarrow$ – S = – a + (a + d) – (a + 2d) + ...+ (a+ (2n – 1)d) – (a + 2nd)

$\therefore$ 2S = a + {– d + d – d + d...upto 2n terms} + (a+2nd)

$\Rightarrow$ 2S = 2a + 2nd S = a + nd **Ans. [C]**

Ex.14 The sum to n terms of the series

$1 + 2\left(1+\frac{1}{n}\right) + 3\left(1+\frac{1}{n}\right)^2 + ...$ is given by-

(A) n^2 (B) n (n+1)

(C) $n(1+1/n)^2$ (D) None of these

Sol. Let S be the sum of n terms of the given series and x = 1 + 1/n, Then,

$S = 1 + 2x + 3x^2 + 4x^3 + ... + n x^{n-1}$

$\Rightarrow xS = x + 2x^2 + 3x^3 + ... + (n-1)x^{n-1} + nx^n$

$\therefore S - xS = 1 + [x + x^2 + ... + x^{n-1}] - nx^n$

$\Rightarrow S(1-x) = \frac{1-x^n}{1-x} - n x^n$

$\Rightarrow S(-1/n) = -n[1-(1+1/n)^n] - n(1+1/n)^n$

$\Rightarrow \frac{1}{n}.S = n[1-(1+1/n)^n + (1+1/n)^n] = n$

$\Rightarrow S = n^2$ **Ans. [A]**

Ex.15 $1 + 2.2 + 3.2^2 + 4.2^3 + ... + 100.2^{99}$ equals-

(A) 99.2^{100} (B) 100.2^{100}

(C) $1 + 99.2^{100}$ (D) None of these

Sol. Let

$S = 1 + 2.2 + 3.2^2 + 4.2^3 + ... + 100.2^{99}$ .....(1)

$\Rightarrow 2S = 2 + 2.2^2 + 3.2^3 + ... + 99.2^{99} + 100.2^{100}$

.....(2)

Subtracting (2) from (1), we get

$-S = (1 + 2 + 2^2 + 2^3 + ... + 2^{99}) - 100.2^{100}$

$\Rightarrow S = 100.2^{100} - \frac{2^{100}-1}{2-1}$

$= 100.2^{100} - 2^{100} + 1$

$= 1 + 99.2^{100}$ **Ans. [C]**

Ex.16 a, b, c are first three terms of a GP. If HM of a and b is 12 and that of b and c is 36, then a equals-

(A) 24 (B) 8 (C) 72 (D) 1/3

Sol. Let given three terms be br, b, b/r

$\therefore\quad 12 = \frac{2(br)b}{br+b} = \frac{2(br)}{r+1}$ ...(1)

and $36 = \frac{2b(b/r)}{b+(b/r)} = \frac{2b}{r+1}$ ...(2)

(1) ÷ (2) $\Rightarrow$ r = 1/3

Then from (2) b = 24

$\therefore$ a = br = 8 **Ans. [B]**

Ex.17 Find three numbers a, b, c between 2 and 18 such that - (i) Their sum is 25 (ii)The numbers 2, a,b are consecutive terms of an A.P. (iii) The numbers b,c, 18 are three consecutive terms of a G.P.

(A) 4, 8, 16 (B) 3, 6, 12

(C) 4, 8, 13 (D) 5, 8, 12

Sol. a + b + c = 25 ...(1)

2, a, b are in A.P. $\Rightarrow$ $a = \frac{2+b}{2}$ ...(2)

and b, c, 18 are in G.P. $\Rightarrow c^2 = 18b$ ...(3)

Eliminating a and b from (1), (2) and (3), gives the following equation $c^2 + 12c - 288 = 0$

(c– 12) (c +24) = 0 c = 12, –24

[Leaving c = – 24 because this is not in between 2 and 18]

$\therefore$ c = 12, from (3) b = 8 and from (1) a = 5

Hence a, b, c = 5, 8,12

Ans. [D]

Ex.18 If x, 1, z are in A.P. x, 2,z are in G.P. then x, 4, z are in-

(A) AP (B) GP

(C) HP (D) None of these

Sol. Here 2 = x + z ...(1)

4 = xz ...(2)

Now $\frac{2xz}{x+z} = \frac{8}{2} = 4$

$\therefore$ x, 4, z are H.P. **Ans. [C]**

Ex.19 If a, b, c in H.P. then value of

$\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right) =$

(A) $\frac{2}{bc}-\frac{1}{b^2}$ (B) $\frac{3}{b^2}-\frac{2}{ab}$

(C) $\frac{3}{ac}-\frac{2}{b^2}$ (D) None of these

Sol. Here a, b, c in H.P. $\frac{2}{b} = \frac{1}{a}+\frac{1}{c}$

Now $\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{2}{b}-\frac{1}{b}\right)$

$=\left(\frac{1}{b}+\frac{2}{b}-\frac{1}{a}-\frac{1}{a}\right)\left(\frac{1}{b}\right) = \frac{3}{b^2}-\frac{2}{ab}$

Also

(eliminating 1/a in first factor and $\frac{1}{c}+\frac{1}{a}$ in second)

$=\left(\frac{2}{c}-\frac{1}{b}\right)\left(\frac{1}{b}\right) = \frac{2}{bc}-\frac{1}{b^2}$

The third one is not an answer **Ans [A, B]**

Ex.20 If $H_1, H_2, H_3, ..., H_n$ be n harmonic means between a and b then $\frac{H_1+a}{H_1-a}+\frac{H_n+b}{H_n-b} =$

(A) 0 (B) n (C) 2n (D) 1

Sol. Here $H_1 = \frac{ab(n+1)}{b(n+1)-(b-a)} = \frac{ab(n+1)}{bn+a}$

Similarly $H_n = \frac{ab(n+1)}{an+b}$ (interchange a and b)

Hence $\frac{H_1+a}{H_1-a}+\frac{H_n+b}{H_n-b}$

$= \frac{(2n+1)b+a}{b-a}+\frac{(2n+1)a+b}{a-b}$

$= \frac{2nb+b+a-2na-a-b}{b-a} = 2n$ **Ans. [C]**

Ex.21 If a,b,c are in H.P. then $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ will be in-

(A) A.P. (B) G.P.

(C) H.P. (D) None of these

Sol. a, b, c are in HP

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP

$\Rightarrow \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are in AP

$\Rightarrow\ 1+\frac{b+c}{a},\ 1+\frac{c+a}{b},\ 1+\frac{a+b}{c}$ are in AP

$\Rightarrow\ \frac{b+c}{a},\ \frac{c+a}{b},\ \frac{a+b}{c}$ are in AP

$\Rightarrow\ \frac{a}{b+c},\ \frac{b}{c+a},\ \frac{c}{a+b}$ are in HP.

Ans.[C]

Ex.22 If the (m+1)th, (n+1)th, (r+1)th terms of an A. P. are in G. P. and m, n, r are in H.P. then the ratio of common difference to the first terms in the A. P. is-

(A) n/2 (B) 2/n (C) – n/2 (D) – 2/n

Sol. Let the first term of A.P. be a and common difference be d.

Given (a + md), (a + nd), (a + rd) in G.P.

$(a + nd)^2 = (a + md)(a + rd)$

$\Rightarrow\ \frac{d}{a} = \frac{2n-m-r}{mr-n^2}$

But m, n, r in H.P. $\Rightarrow\ n = \frac{2mr}{m+r}$

$\therefore\ \frac{d}{a} = \frac{2n-\frac{2mr}{n}}{mr-n^2} = \frac{2}{n}\left(\frac{n^2-mr}{mr-n^2}\right) = -\frac{2}{n}$

Ans. [D]

Ex.23 If d, e, f are in G.P. and two quadratic equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root then, d/a, e/b, f/c are in-

(A) H. P. (B) G. P.

(C) A. P. (D) None of these

Sol. Here $e^2 = df$

Now $dx^2 + 2ex + f = 0$ given

$\Rightarrow\ dx^2 + 2\sqrt{df}\ x + f = 0 \Rightarrow\ x = -\sqrt{\frac{f}{d}}$

Putting in $ax^2 + 2bx + c = 0$ we get

$a\frac{f}{d} + c = 2b\sqrt{\frac{f}{d}}$

$\Rightarrow\ \frac{a}{d} + \frac{c}{f} = \frac{2b}{e}$

$\therefore\ \frac{a}{d}, \frac{b}{e}, \frac{c}{f}$ are in A.P.

$\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in H.P. **Ans. [A]**

Ex.24 If a, b, c in A.P. and $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$, $z = \sum_{n=0}^{\infty} c^n$ then x, y, z are in-

(A) AP (B) GP

(C) HP (D) None of these

Sol. Here a, b, c in A.P. , given

Also $x = \frac{1}{1-a}$, $y = \frac{1}{1-b}$, $z = \frac{1}{1-c}$

Now a, b, c in AP

$\Rightarrow$ 1 – a, 1 – b, 1 – c in A. P.

$\Rightarrow\ \frac{1}{1-a}, \frac{1}{1-b}, \frac{1}{1-c}$ in H. P.

$\Rightarrow$ x, y, z in H. P **Ans. [C]**

Ex. 25 If a, x, y, z, b are in AP, then x+ y + z = 15 and if a, x, y, z, b are in HP, then $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{5}{3}$. Numbers a, b are-

(A) 8, 2 (B) 11, 3

(C) 9, 1 (D) None of these

Sol. By property of A.P. x + z = a + b and y = 1/2 (a + b)

$\Rightarrow\ x + y + z = \frac{3}{2}(a + b)$

$\Rightarrow\ a + b = 10$ ...(1)

Also $\frac{1}{a}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{b}$ are in AP, so as above

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{3}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$

$\Rightarrow\ \frac{1}{a}+\frac{1}{b} = \frac{10}{9}$

$\Rightarrow\ ab = 9$ ...(2)

From (1) and (2) a, b are 9, 1 **Ans. [C]**

Ex.26 If rth term of a series is $(2r + 1)2^{-r}$, then sum of its infinite terms is-

(A) 10 (B) 8 (C) 5 (D) 0

Sol. Here $T_r = (2r + 1)2^{-r}$

$\therefore$ Series is : $\frac{1}{2}\left[3+\frac{5}{2}+\frac{7}{2^2}+....\right]$

Obviously the series in the bracket is Arithmetico-Geometrical Series. Therefore by the formula

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

We have

$$S_\infty = \frac{1}{2}\left[\frac{3}{1-\frac{1}{2}} + \frac{2\left(\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)^2}\right] = 5$$

Ans. [C]

Ex.27 $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is AM/GM/HM, between a and b if n is equal to respectively-

(A) $-1, -\frac{1}{2}, 0$ (B) $0, \frac{1}{2}, -\frac{1}{2}$

(C) $0, -\frac{1}{2}, -1$ (D) None of these

Sol. By trial,

putting n = 0, $\frac{a^{0+1}+b^{0+1}}{a^0+b^0} = \frac{a+b}{2}$ = A.M.

Putting $n = -\frac{1}{2}$, $\frac{a^{-\frac{1}{2}+1}+b^{-\frac{1}{2}+1}}{a^{-\frac{1}{2}}+b^{-\frac{1}{2}+1}} = \frac{\sqrt{a}+\sqrt{b}}{\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}}$

$= \sqrt{ab}$ = G.M.

$n = -1$, $= \frac{a^0+b^0}{a^{-1}+b^{-1}} = \frac{2ab}{a+b}$

H. M. Hence, option (C) is correct. **Ans. [C]**

Alternately

for AM

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{a+b}{2}$$

$\Rightarrow 2a^{n+1} + 2b^{n+1} = a^{n+1} + b^{n+1} + a^nb + ab^n$

$\Rightarrow a^{n+1} - a^nb = -b^{n+1} + ab^n$

$\Rightarrow a^n(a-b) = +b^n(a-b)$, $a \neq b$

$\Rightarrow \left(\frac{a}{b}\right)^n = 1$

$\Rightarrow$ n = 0, similarly for GM and HM also.

Ex.28 If $a_1, a_2, a_3, \ldots, a_n$ are in HP, then $a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n$ is equal to-

(A) na_1a_n (B) $(n-1)a_1a_n$

(C) $(n+1)a_1a_n$ (D) None of these

Sol. Let d be common difference of the corresponding AP.

So $\frac{1}{a_2} - \frac{1}{a_1} = d, \frac{1}{a_3} - \frac{1}{a_2} = d, \ldots, \frac{1}{a_n} - \frac{1}{a_{n-1}} = d$

$\Rightarrow a_1 - a_2 = d(a_1a_2)$, $a_2 - a_3 = d(a_2a_3)$,...., $(a_{n-1} - a_n) = d(a_{n-1}a_n)$

Adding these relations, we get

$a_1 - a_n = d(a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n)$ ...(1)

Also $\frac{1}{a_n} = T_n = \frac{1}{a_1} + (n-1)d$

$\Rightarrow \frac{1}{a_n} - \frac{1}{a_1} = (n-1)d$

$\Rightarrow a_1 - a_n = (n-1)d(a_1a_n)$ ...(2)

From (1) and (2), we have

$(n-1)(a_1a_n) = a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n$

Ans. [B]

Ex.29 Sum of the series 3 + 7 + 14 + 24 + 37 +...10 terms, is –

(A) 560 (B) 570

(C) 580 (D) None of these

Sol. Here the given series is not A.P., G.P., or HP

Let $S = 3 + 7 + 14 + 24 + 37 + \ldots + T_n$

$S = 3 + 7 + 14 + 24 + \ldots + T_n$

after subtracting

$0 = 3 + \underbrace{4+7+10+13+\ldots - T_n}_{\text{A.P.}}$

$\therefore T_n = 3 + \frac{(n-1)}{2}[2(4) + (n-2)3]$

$= 1/2\ (3n^2 - n + 4)$

$\therefore \quad S_n = 1/2\ [3\Sigma n^2 - \Sigma n + 4n]$

$= 1/2 \left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right]$

Putting n = 10

$S_{10} = 1/2 \left[\frac{10\times11\times21}{2} - \frac{10\times11}{2} + 40\right]$

$= 1/2\ [1155 - 55 + 40] = \frac{1140}{2} = 570$

Ans. [B]

Note : First apply the method of difference for n terms and proceed.

Ex.30 $2^{1/4}.\ 2^{2/8}.\ 2^{3/16}.\ 2^{4/32}......\infty$ is equal to-

(A) 1 (B) 2 (C) 3/2 (D) 5/2

Sol. The given product

$= 2^{\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+...} = 2^S$ (say)

Now $S = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + ...$.

..(1)

$\Rightarrow \quad \frac{1}{2} S = \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + ...$

...(2)

(1) – (2)

$\Rightarrow \quad \frac{1}{2} S = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$

$= \frac{1/4}{1-\frac{1}{2}} = 1/2 \qquad \therefore \quad S = 1$

$\Rightarrow$ Product $= 2^1 = 2$ **Ans. [B]**